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Ludovic Renou

School of Economics
University of Adelaide University, 5005 Australia

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Ludovic Renou*

University of Adelaide

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Abstract

A social game is a generalization of a strategic-form game, in which not only the payoff of each player depends upon the strategies chosen by their opponents, but also their set of admissible strategies. Debreu (1952) proves the existence of a Nash equilibrium in social games with continuous strategy spaces. Recently, Polowczuk and Radzik (2004) have proposed a discrete counterpart of Debreu's theorem for two-person social games satisfying some "convexity properties". In this note, we define the class of supermodular social games and give an existence theorem for this class of games.

KEYWORDS: Strategic-form games, social games, supermodularity, Nash equilibrium, existence.

JEL Classification Numbers: C7.

*Department of Economics, Room G52, Napier Building, Adelaide 5005, Australia. Phone: +61 (0)8 8303 4930, Fax: +61 (0)8 8223 1460. `ludovic.renou@adelaide.edu.au`

1 Introduction

A ***social game*** is a generalization of a strategic-form game, in which the set of "admissible" strategies of a player is constrained by the strategies of the other players. Historically, Arrow and Debreu (1954) were the first to introduce the concept of a social game, calling it an *abstract economy* in their original paper. To motivate the need for this generalization, Arrow and Debreu invoked the special position of consumers in an economy. The strategies of a consumer can be regarded as the choice of different bundles of goods. Theirs is a constrained choice in that the total cost of the goods chosen at market prices cannot exceed their disposable income. In turn, market prices and the disposable income are determined by the choices of other agents in the economy e.g., tax authorities or employers.

Formally, a social game G is a tuple $< N, (X_i, u_i, S_i)_{i \in N} >$. $N = \{1, \ldots, n\}$ is the set of players, X_i is the set of pure strategies available to player i. Denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$, and x_{-i} an element of X_{-i}. For each player $i \in N$, S_i is a multi-valued map from the set X_{-i} to subsets of the set X_i, with $S_i(x_{-i})$ the set of pure strategies admissible to player i when his opponents play x_{-i}. Hence, the map S_i represents the social constraint imposed by player i's opponents on his behavior. Player i's payoff function is $u_i : X_i \times X_{-i} \to \mathbb{R}$. The *mixed extension* of a finite social game G is the tuple $< N, (\Delta(X_i), v_i, S_i)_{i \in N} >$ where $\Delta(X_i)$ is the set of probabilities on X_i, for all $(p_i, p_{-i}) \in \prod_{i \in N} \Delta(X_i)$, $v_i(p_i, p_{-i}) = \sum_{x_i \in X_i} \sum_{x_{-i} \in X_{-i}} u_i(x_i, x_{-i}) p_i(x_i) p_{-i}(x_{-i})$, and

$$S_i(p_{-i}) = \bigcap_{x_{-i} \in \text{ supp } p_{-i}} S_i(x_{-i}).$$

A profile of strategies $x^* = (x_i^*, x_{-i}^*)$ is an Arrow-Debreu-Nash equilibrium, (hereafter, an equilibrium), of the social game G if for each player $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and

$$x_i^* \in BR_i(x_{-i}^*) := \arg \max_{x_i \in S_i(x_{-i}^*)} u_i(x_i, x_{-i}^*).$$

A profile of strategies is a mixed equilibrium of G if it is a equilibrium of the mixed extension of G. Formally, (p_i^*, p_{-i}^*) is a mixed equilibrium of G if for each player $i \in N$, supp $p_i^* \subset S_i(p_{-i}^*)$, and $p_i^* \in BR_i(p_{-i}^*)$. This note is concerned with the equilibrium existence in social games.

In "Social equilibrium existence theorem" (1952), Debreu provides sufficient conditions for the existence of an equilibrium in a social game. The sufficient conditions for existence are as follows. For each player $i \in N$, X_i is a contractible polyhedron, S_i is a semi-continuous multi-valued map (i.e., its graph is closed), u_i is continuous, and for each x_{-i}, the set $S_i(x_{-i})$ is contractible. (See Theorem, p. 888 in Debreu (1952).) In an historical note, Debreu also mentions the existence of an equilibrium if for each player $i \in N$, X_i is a non-empty, compact, convex subset of a finite Euclidean space, u_i is continuous and quasi-concave in x_i, and the multi-valued map S_i is semi-continuous, non-empty and convex-valued. This last statement is now familiar to game theorists as it relies on the celebrated Kakutani fixed-point theorem.

Recently, Polowczuk and Radzik (2004) have provided a counterpart of Debreu's theorem for two-player non-zero sum games with finite strategy spaces. Their main assumptions are: Z1) symmetry i.e., for all $x_2 \in S_2(x_1)$, $x_1 \in S_1(x_2)$ and for all $x_1 \in S_1(x_2)$, $x_2 \in S_2(x_1)$, Z2) sections convexity i.e., a discrete counterpart of the convex-valuedness of S_i, and Z3) game convexity i.e., a discrete counterpart of the quasi-concavity of u_i. Assuming Z1-Z3, they prove the existence of an equilibrium in (mixed) strategies consisting of two *two-adjoining pure strategies* i.e., mixed strategies assigning strictly positive probability to only two consecutive pure strategies x and $x+1$. However, their theorem (Theorem 4) does not hold true for three players or more. For instance, consider the following three-player game (Figure 1).

	0	1
0	$1,1,0$	$\star,\star,\star$
1	$0,1,1$	$1,0,1$

0

	0	1
0	$1,0,1$	$0,1,1$
1	$0,0,0$	$1,1,0$

1

Figure 1: A three-player social game with no equilibrium.

Each player has two strategies 0 and 1. Player 1 chooses a row, player 2 a column and player 3 a matrix. Moreover, the profile of strategies $(0,1,0)$

is not admissible i.e., we have $S_1(1,0) = \{1\}$, $S_2(0,0) = \{0\}$, $S_3(0,1) = \{1\}$ and for each player i, $S_i(x_{-i}) = \{0,1\}$, otherwise. We can easily check that this game has no Nash equilibrium.

In the next section, we define the class of supermodular social games and prove the existence of a Nash equilibrium. In particular, our result holds for n-player games with finite unidimensional strategy spaces if payoffs have increasing differences.

2 Supermodular social games and equilibrium existence

The following definition of a supermodular social game generalizes the definition of a supermodular game introduced in Milgrom and Roberts (1990). Both definitions coincide when there is no social constraints i.e., if for all players, for all $x_{-i} \in X_{-i}$, $S_i(x_{-i}) = X_i$. We refer the reader to Topkis (1998) for the definitions of concepts introduced below.

Definition 1 *A social game* $G =< N, (u_i, X_i, S_i)_{i \in N} >$ *is* ***supermodular****, if for each player* $i \in N$*,*

(A1) X_i *together with the order* $\geq_i$ *is a non-empty complete lattice;*

(A2) $u_i : X \to \mathbb{R} \cup \{-\infty\}$ *is order upper semi-continuous in* x_i *(for a fixed* x_{-i}*), order-continuous in* x_{-i} *(for fixed* x_i*), and has a finite upper-bound;*

(A3) u_i *is supermodular in* x_i *(for fixed* x_{-i}*);*

(A4) u_i *has increasing differences in* x_i *and* x_{-i}*;*

(A5) S_i *is ascending in* x_{-i}*, and, for each* x_{-i}*,* $S_i(x_i)$ *is a non-empty complete sublattice of* X_i*.*

Conditions (A1)-(A4) are equivalent to conditions (A1)-(A4) of Milgrom and Roberts (1990). However, condition (A5) has no equivalence in Milgrom and Roberts as they do not consider social games. The first part of (A5) together with (A3)-(A4) insures that player i's best-reply map BR_i is non-decreasing on the set $\{x_{-i} \in X_{-i} : BR_i(x_{-i}) \neq \emptyset\}$. The second part of (A5) together with (A1)-(A2) insures that best-reply maps are everywhere non-empty valued. The following example illustrates the importance of this condition. Let $\overline{\mathbb{R}}$ be the extended real line i.e., $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Together with the usual order $\geq$, $\overline{\mathbb{R}}$ is a complete lattice. Consider the subset $S = \{-2\} \cup (-1, +1) \cup \{+2\}$ of $\overline{\mathbb{R}}$. S is a complete lattice, a sublattice of the extended real line, but not a complete sublattice as $\sup_S(0,1) = 2 \neq 1 = \sup_{\overline{\mathbb{R}}}(0,1)$. It is then easy to see that the function $x \mapsto f(x) = -x^2 + 2x$ is *order continuous* on $\overline{\mathbb{R}}$, while it is not *order upper semi-continuous* on S; and f has no maximum in S, while it has a maximum in $\overline{\mathbb{R}}$.

Lemma *If f is an order upper semi-continuous, supermodular function on a complete lattice X, then any restriction of f to a complete sublattice S of X is order upper semi-continuous and supermodular on S.*

Proof Let $C \subseteq S$ be a chain, i.e., a totally ordered subset of S. Note that C is also a chain of X. Since S is a complete sublattice of X, we have that $\sup_S(C) = \sup_X(C)$. It follows that

$$\lim \sup_{x \in C,\, x \uparrow \sup_S(C)} f(x) = \lim \sup_{x \in C,\, x \uparrow \sup_X(C)} f(x) \leq f(\sup_X(C)) = f(\sup_S(C)),$$

since f is order upper semi-continuous on X. A similar reasoning holds for the convergence to $\inf_S(C)$, hence f is order upper semi-continuous on S. Finally, it is trivial to prove that f is supermodular on S. □

We can now state our main theorem, which proves the existence of an equilibrium in supermodular social games.

Theorem *A supermodular social game has an equilibrium.*

Proof Fix a $x_{-i} \in X_{-i}$. Since $S_i(x_{-i})$ is a complete sublattice of X_i, hence a complete lattice in its own right, and u_i is supermodular and order upper semi-continuous on $S_i(x_{-i})$ by Lemma, a direct application of Theorem 1 (p1262) of Milgrom and Roberts (1990) proves the existence of a maximum of $u_i(\cdot, x_{-i})$ in $S_i(x_{-i})$. It follows that $BR_i(x_{-i}) \neq \emptyset$ for all $x_{-i} \in X_{-i}$. From Theorem 6.1 in Topkis (1978), we have that BR_i is ascending in x_{-i} on the set $\{x_{-i} : BR_i(x_{-i}) \neq \emptyset\}$. Existence of a Nash equilibrium then follows by Tarski fixed-point theorem as in Topkis (1979). □

As a final remark, it is worth noting that a similar result can already be found in Topkis (1979, Theorem 3.1 (p. 781)), although it seems that Topkis did not realize the relation of his result with social games. Moreover, our theorem slightly improves upon Topkis' theorem as Topkis considers compact intervals of finite Euclidean spaces for strategy spaces and continuous payoff functions, while we consider more general strategy spaces and payoff functions.

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